4.5 - Solving Polynomial Equations

Warm-up - Identify graphs with functions

a. $x^3 - 6x^2 + 12x - 8 = 0$ **D**

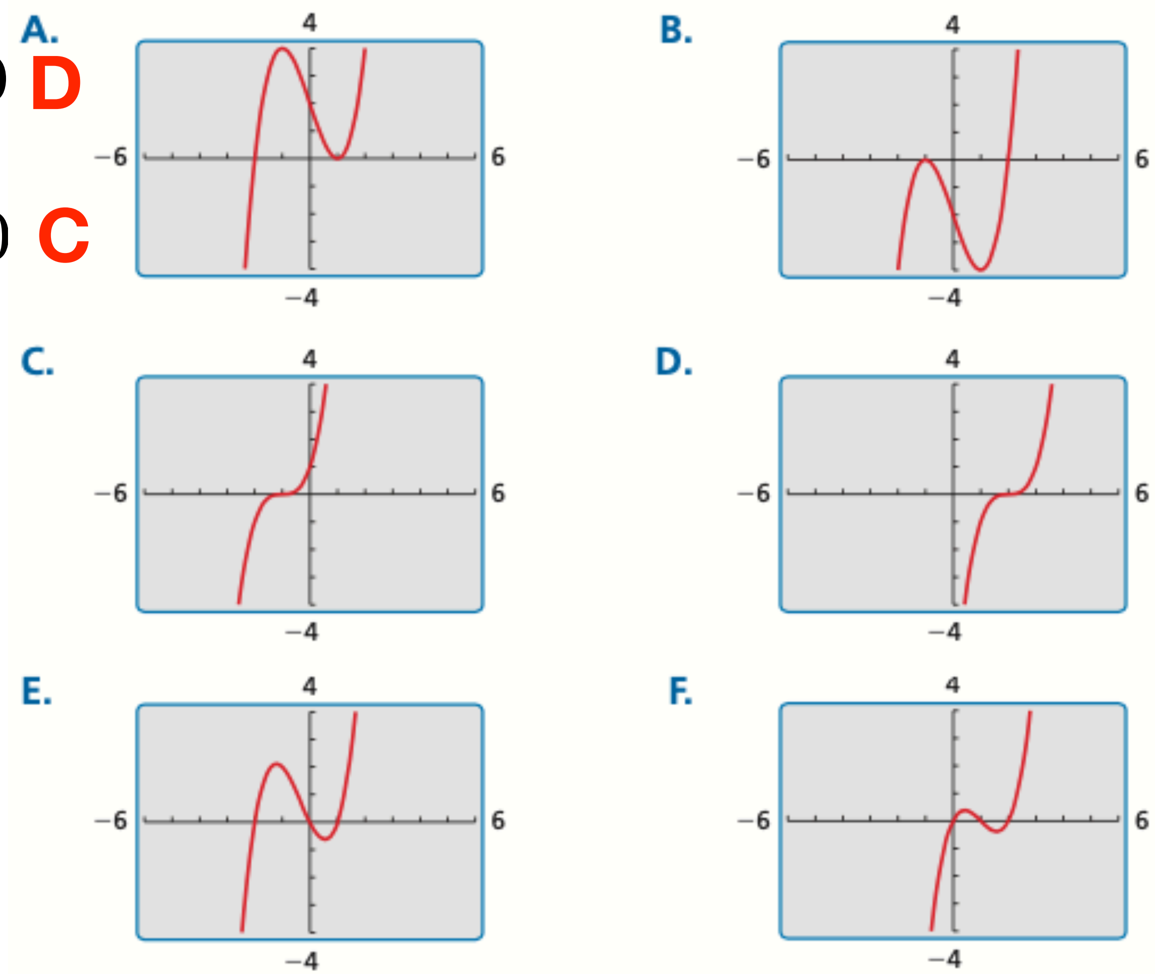
b. $x^3 + 3x^2 + 3x + 1 = 0$ **C**

c. $x^3 - 3x + 2 = 0$ **A**

d. $x^3 + x^2 - 2x = 0$ **E**

e. $x^3 - 3x - 2 = 0$ **B**

f. $x^3 - 3x^2 + 2x = 0$ **F**



4.5 - Solving Polynomial Equations

Solve by Factoring

$$5x^3 - 40x^2 + 80x = 0$$

Roots: 0, 4, 4

$$x^4 - 18x^2 + 81 = 0$$

Roots: 3, 3, -3, -3

4.4 - Factoring Polynomials

Factor In Quadratic Form

1. $x^{8a+8b} - y^{16c-8d}$

$\left(x^{4a+4b} + y^{8c-4d}\right)\left(x^{2a+2b} + y^{4c-2d}\right)\left(x^{a+b} + y^{2c-d}\right)\left(x^{a+b} - y^{2c-d}\right)$

2. $16x^4 - 8x^2 + 1$

$(2x-1)^2(2x+1)^2$

4.5 - Solving Polynomial Equations

Solve given one root

1. $x^3 - 2x^2 - x + 2 = 0; -1$

Roots: -1, 1, and 2

2. $2y^3 - 5y^2 - 4y + 3 = 0;\ 3$

Roots: -1, $\frac{1}{2}$, and 3

3. $2k^3 + \sqrt{3}k^2 - 15k + 6\sqrt{3} = 0;\ \sqrt{3}$

Roots: $\sqrt{3}$, $-2\sqrt{3}$, and $\frac{\sqrt{3}}{2}$

4.5 - Solving Polynomial Equations

The Rational Root Theorem

If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has *integer* coefficients, then every rational solution of $f(x) = 0$ has the following form:

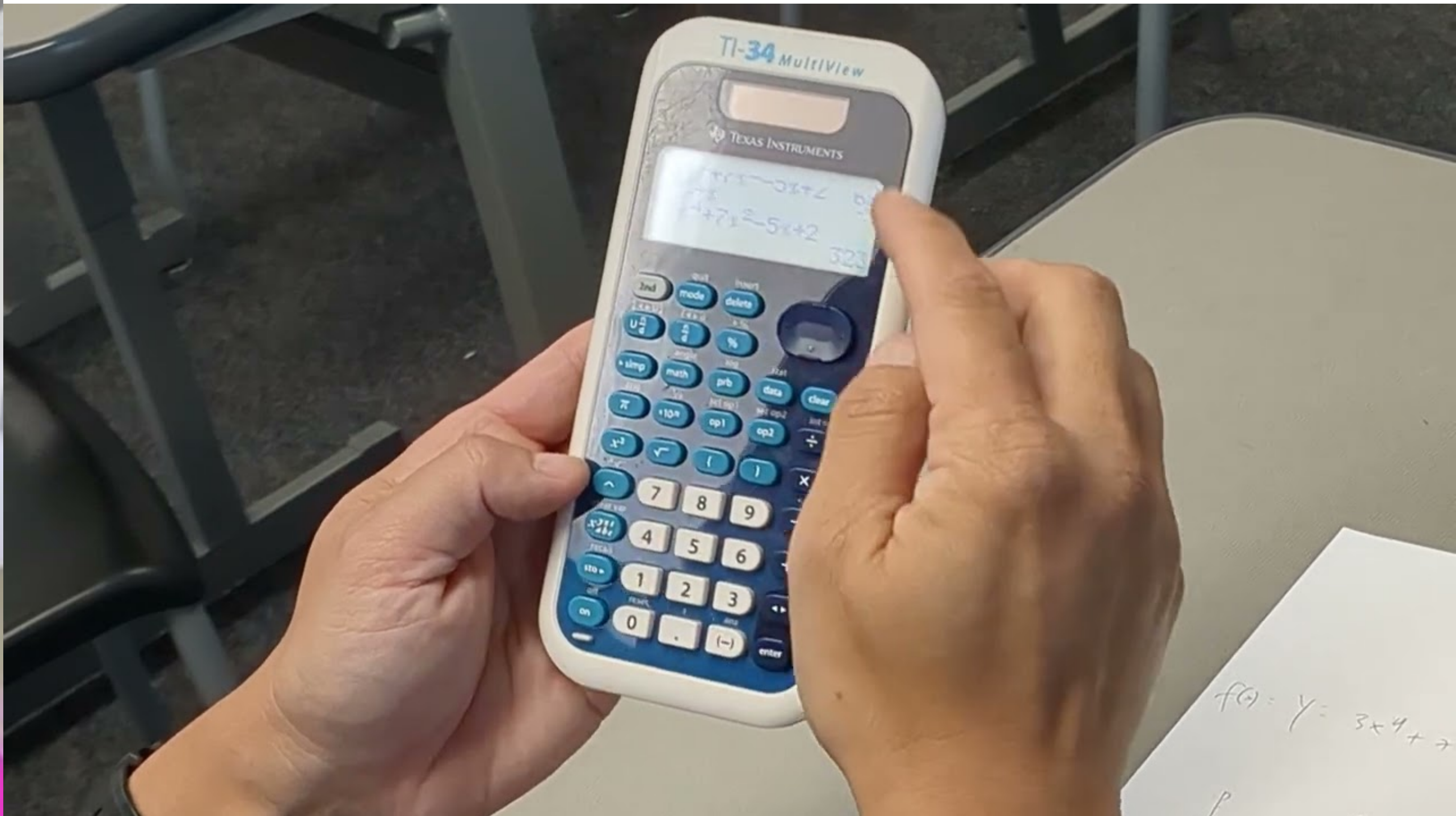
$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

For example $2x^4 + 3x^3 - 7x^2 + 3x - 9 = 0$

Possible rational roots

$$x = \pm 1, \pm 3, \pm 9, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}$$

Actual roots $x = \{-3, \frac{3}{2}, i, -i\}$



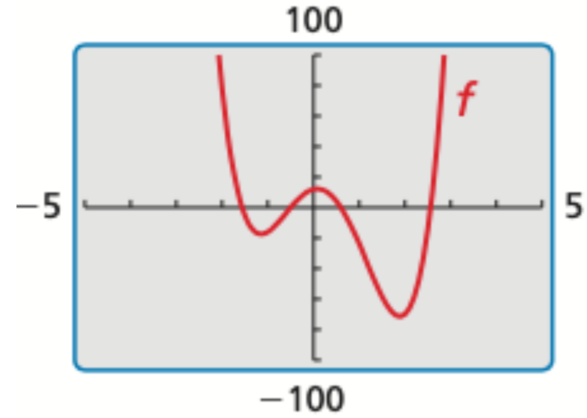
Testing roots

on a Calculator

4.5 - Solving Polynomial Equations

Find Zeros of Polynomial Function

$$f(x) = 10x^4 - 11x^3 - 42x^2 + 7x + 12$$



$$x = \{-\frac{1}{2}, \frac{3}{5}, \frac{1 \pm \sqrt{17}}{2}\}$$

4.5 - Solving Polynomial Equations

Find Zeros of Polynomial Function

$$f(x) = 4x^4 - 4x^3 + 29x^2 + 34x + 9$$

Hint: Roots are negative

$$x = \{-\frac{1}{2},\ -\frac{1}{2},\ 1 \pm 2i\sqrt{2}\}$$

4.6 - The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra

Theorem If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has at least one solution in the set of complex numbers.

Corollary If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has exactly n solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

Equation	Roots	Degree of equation Number of roots
$x + 1 = 0$	-1	1
$x^2 - 4x + 13 = 0$	$2 + 3i,\ 2 - 3i$	2
$x^3 + 4x^2 + 4x = 0$	$0,\ -2,\ -2$	3
$x^4 - 10x^2 + 9 = 0$	$1,\ -1,\ 3,\ -3$	4
$x^5 - 16x = 0$	$0,\ 2,\ -2,\ 2i,\ -2i$	5

4.5 - Solving Polynomial Equations

The Irrational Conjugates Theorem

Let f be a polynomial function with rational coefficients, and let a and b be rational numbers such that $\sqrt{b}$ is irrational. If $a + \sqrt{b}$ is a zero of f, then $a - \sqrt{b}$ is also a zero of f.

For example

Write a polynomial function f to least degree that has rational coefficients, a leading coefficient of 1, and the zeros 3 and $2 + \sqrt{5}$

$$f(x) = x^3 - 7x^2 + 11x + 3$$

4.5 - Solving Polynomial Equations

Conjugate Root Theorem

If a+bi is a root
then a-bi is a root.

$$x^2 + 2x + 5 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{-16}}{2}$$

$$x = -1 \pm 2i$$

4.5 - Solving Polynomial Equations

Conjugate Root Theorem

1. Factor $x^4 - 12x - 5 = 0$ given that $-1 + 2i$ is a root.

$$x = \{-1 \pm 2i,\ 1 \pm \sqrt{2}\}$$

4.5 - Solving Polynomial Equations

Conjugate Root Theorem

2. Factor $f(x) = x^4 + 3x^3 - 3x^2 + 3x - 4$ given that i is a root.

$x = \{-4,\ 1,\ \pm i\}$

4.5 - Solving Polynomial Equations

Conjugate Root Theorem

Find polynomial equation if roots are 2 and $3 - i$

$$(x-2)(x-(3-i))(x-(3+i))$$

$$(x-2)((x-3)+i))((x-3)-i))$$

$$(x-2)((x-3)^2-i^2))$$

$$x^3-6x^2+10x-2x^2+12x-20$$

$$x^3-8x^2+22x-20=0$$

4.5 - Solving Polynomial Equations

Conjugate Root Theorem

1. Find polynomial equation if roots are -1 and $5 + 2i$

$$x^3 - 9x^2 + 19x + 29 = 0$$

4.6 - The Fundamental Theorem of Algebra

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial function with real coefficients.

- The number of *positive real zeros* of f is equal to the number of changes in sign of the coefficients of $f(x)$ or is less than this by an even number.
- The number of *negative real zeros* of f is equal to the number of changes in sign of the coefficients of $f(-x)$ or is less than this by an even number.

How many positive and negative real zeros does the function have? How many imaginary zeros?

$$f(x) = x^6 + x^5 - 17x^4 - 15x^3 + 14x^2 - 16x + 32$$

4.6 - The Fundamental Theorem of Algebra

Decartes's Rule of Signs

$$P(x) = x^5 - 4x^4 - 6x^3 + 6x^2 - 7x + 10$$

$$P(-x) = -x^5 - 4x^4 + 6x^3 + 6x^2 + 7x + 10$$

+ Real Roots	- Real Roots	Img Roots	Total Roots

4.6 - The Fundamental Theorem of Algebra

Decartes's Rule of Signs

Positive = 5 changes

Negative = 2 changes

Total = degree 9

+ Real Roots	- Real Roots	Img Roots	Total Roots